Modern approaches to quantum gravity

Solution 2 Fall 2025

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1. Mechanics of spherically symmetric black holes

(a) Neglecting, the S^2 factor, the geometry near the horizon is

$$ds^2 = f'(r_h)(r - r_h)d\tau^2 + \frac{dr^2}{f'(r_h)(r - r_h)} \tag{1}$$

Using a transformation $r - r_h = \rho^2$, we get

$$ds^2 = \frac{4}{f'(r_h)}\Big(\frac{(f'(r_h))^2}{4}\rho^2 d\tau^2 + d\rho^2\Big) \tag{2}$$

This is regular at $\rho = 0$ if the manifold is locally isomorphic to $\mathbb{R}^2$, hence we require $\tau \sim \tau + \frac{4\pi}{|f'(r_h)|}$.

(b) The relation between surface gravity and temperature is fixed by black hole thermodynamics $\kappa = 2\pi T_h$, and the interpretation of the area of a black hole as its entropy $S = \frac{A}{4}$. One matches the first laws

$$\begin{aligned} \frac{\kappa}{8\pi G} dA &= dM + ... \qquad (3) \\ T dS &= dE + ... \qquad (4) \end{aligned}$$

Let us find now the surface gravity with a geometric approach. In EF coordinates we get

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{5}$$

If the metric is asymptotically flat, then the vector $\xi = \partial_u$ satisfies our requirements: $\xi^2 = -f(r) \to -1$ as $r \to \infty$ and it is a null Killing vector at $r = r_h$. Therefore, we get

$$\nabla_\mu(\xi^2)|_{r_h} = -f'(r_h)\delta^r_\mu \tag{6}$$

Using $\xi_\mu|_{r_h} = g_{\mu v}|_{r_h} = \delta^r_\mu$, we obtain the surface gravity

$$\kappa = \frac{|f'(r_h)|}{2} \tag{7}$$

(c) The black hole has an horizon at $r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$, with $r_h \equiv r_+$ being the outer one.
If we write $f(r) = \frac{(r - r_-)(r - r_+)}{r^2}$, we easily find the temperature is

$$T = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{G^2M^2 - GQ^2}}{2\pi(2G^2M^2 - GQ^2 + 2GM\sqrt{G^2M^2 - GQ^2})} \tag{8}$$

For Schwarzschild, we recover $T = \frac{1}{8\pi MG}$.

(d) The area of the black hole is simply

$$A = 4\pi r_+^2 = 4\pi(2G^2M^2 - GQ^2 + 2GM\sqrt{G^2M^2 - GQ^2}) \tag{9}$$

For Schwarzschild, we recover $A = 16\pi G^2M^2$.

(e) The electric potential is given by:

$$\Phi = -\frac{T}{4G}\frac{\partial A}{\partial Q} \tag{10}$$

(f) Before the collision, the total area of the two black holes is $A_i = 16\pi G^2(M_1^2 + M_2^2)$. After the collision, it is $A_f = 16\pi G^2M_3^2$.
The second law implies $A_f \geq A_i$, hence $M_3 \geq M_{cr} = \sqrt{M_1^2 + M_2^2}$.

2. Raychaudhuri equation

(a) Since we assume $\hat{\omega}_{\mu\nu} = 0$ and since $\hat{\sigma}_{\mu\nu}$ is spacelike, we have that

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu \leq -\frac{1}{2}\theta^2 - R_{\mu\nu}k^\mu k^\nu \tag{11}$$

Using the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{12}$$

One can contract with $g^{\mu\nu}$ to obtain

$$R = -8\pi G T \tag{13}$$

where $T \equiv T_{\mu\nu}g^{\mu\nu}$. Plugging this back into the original Einstein's equation, we get

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{14}$$

Now contracting with $k^\mu k^\nu$ using that $k^\mu k_\mu = 0$ and the null energy condition, we obtain

$$R_{\mu\nu}k^\mu k^\nu = 8\pi G T_{\mu\nu}k^\mu k^\nu \geq 0 \tag{15}$$

Thus,

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \tag{16}$$

Rearranging,

$$d\left(\frac{1}{\theta}\right) \geq \frac{1}{2}d\lambda \tag{17}$$

Integrating between $(0, \theta_0)$ and (λ_0, θ_f), we get

$$\frac{1}{\theta_f} - \frac{1}{\theta_0} \geq \frac{\lambda_0}{2} \tag{18}$$

taking $\theta_f \to \infty$ and θ_0 negative, we obtain the suggested bound,

$$\lambda_0 \leq \frac{2}{|\theta_0|} \tag{19}$$

(b) As the hint suggests, we need to verify $\xi^\alpha \nabla_\alpha \xi^\mu = 0$. Since ξ^μ only has a V component, this reduces to

$$0 = \xi^V \nabla_V \xi^\mu = \xi^V \partial_V \xi^\mu + \Gamma^\mu_{VV} \xi^V \tag{20}$$

Using

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\nabla_\mu g_{\nu\sigma} + \nabla_\nu g_{\mu\sigma} - \nabla_\sigma g_{\mu\nu}) , \tag{21}$$

we get that that the only non-trivial component of Γ^μ_{VV} is

$$\Gamma^V_{VV} = g^{-1}_{UV} \partial_V (g_{UV}) \tag{22}$$

where g_{UV} is the component of the metric

$$g_{UV} = -\frac{16M^3 e^{-r/(2M)}}{r} \tag{23}$$

To compute $\partial_V \xi^V$, we need $\partial_V r$. This is computed by deriving the implicit relation $r = r(U, V)$ on both sides with respect to V. We obtain

$$\partial_V r = -U \frac{4M^2}{r} e^{-r/(2M)} \tag{24}$$

Plugging everything and computing the derivatives, we obtain the desired result.

(c) Using the hint,

$$\theta = r^{-1} e^{r/(2M)} \partial_V (\underbrace{r e^{-r/(2M)}}_{\sqrt{-g}} r e^{r/(2M)}) = 2 e^{r/(2M)} \partial_V r \tag{25}$$

Using the result obtained previously for $\partial_V r$, we get

$$\theta = -\frac{8M^2}{r} U \tag{26}$$

The shear and twist do not appear for the following reasons. The shear measures how, following a test of cloud particles across the geodesic congruence, the shape of the cloud changes. Here, since the geodesics are radial, they only see transversely the round 2-sphere which has no preferred direction, so the shape cannot change, it remains spherically symmetric and the shear vanishes. The twist measures how two nearby geodesics "corkscrew", or spin around each other. Note that it vanishes whenever the geodesic congruence is everywhere orthogonal to an hypersurface, since following to normal of hypersurfaces leaves no room for "corkscrewing". Here, since ξ^μ is everywhere orthogonal to the hypersurfaces of constant V, we conclude that the twist vanishes. This can also be checked by explicit computations.

(d) Using the chain rule,

$$\frac{d\theta}{d\lambda} = \frac{dV}{d\lambda} \frac{d\theta}{dV} \tag{27}$$

and using $dV/d\lambda = \xi^V = r e^{r/(2M)}$ by definition of ξ^μ, taking the derivative $d\theta/dV$ using the expression of θ and the formula for $\partial_V r$, we obtain

$$\frac{d\theta}{d\lambda} = \underbrace{-\frac{32M^4}{r^3} U e^{-r/(2M)}}_{d\theta/dV} r e^{r/(2M)} = -\frac{1}{2} \theta^2 \tag{28}$$

(e) Follow the discussion in point (a) by replacing $\leq$, $\geq$ by $=$. Our situation thus corresponds to the upper bound $\lambda_0 = 2/|\theta_0|$.